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Radially infalling brane and moving domain wall in the brane cosmology

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Abstract

We discuss the brane cosmology in the 5D anti de Sitter Schwarzschild ($AdSS_5$) spacetime. A brane with the tension σ is defined as the edge of an $AdSS_5$ space. We point out that the location of the horizon is an apparently, singular point at where we may not define an embedding of the $AdSS_5$ spacetime into the moving domain wall (MDW). We resolve this problem by introducing a radially infalling brane (RIB) in $AdSS_5$ space, where an apparent singularity turns out to be a coordinate one. Hence the CFT/FRW-cosmology is well-defined at the horizon. As an example, an universal Cardy formula for the entropy of the CFT can be given by the Friedmann equation at the horizon.

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I. INTRODUCTION

Recently there has been much interest in the phenomenon of localization of gravity proposed by Randall and Sundrum (RS) [1,2]. RS assumed a single positive tension 3-brane and a negative bulk cosmological constant in the 5D spacetime [2]. They have obtained a 4D localized gravity by fine-tuning the tension of the brane to the cosmological constant. More recently, several authors have studied its cosmological implications. The brane cosmology contains some important deviations from the Friedmann-Robertson-Walker (FRW) cosmology. One approach is first to assume the 5D dynamic metric (that is, BDL-metric [3,4]) which is manifestly Z_2-symmetric. Then one solves the Einstein equation with a localized stress-energy tensor to find the behavior of the scale factor. We call this the BDL approach.

The other Z_2-symmetric approach starts with a static configuration which is two AdSS$_5$ spaces joined by the domain wall. In this case the embedding into the moving domain wall[1] is possible by choosing a normal vector n_M and a tangent vector u_M [5–7]. The domain wall separating two such bulk spaces is taken to be located at $r = a(\tau)$, where $a(\tau)$ will be determined by solving the Israel junction condition [8]. Then an observer on the wall will interpret his motion through the static bulk background as cosmological expansion or contraction [9].

On the other hand, brane cosmology has been studied in the AdS/CFT correspondence. For example, the holographic principle was investigated in a FRW universe with a conformal field theory (CFT) within an AdSS$_5$-bulk theory [10]. In this case the brane is considered as the edge of an AdSS$_5$ space. The brane starts with (big bang) inside the small black hole[2] ($\ell > r_+$), crosses the horizon, and expands until it reaches maximum size. And then the brane contracts , it falls the black hole again and finally disappears (big crunch). An observer in AdSS$_5$-space finds two interesting moments (two points in the Penrose diagram [11]) when the brane crosses the past (future) event horizon. Authors in ref. [12] insisted that at these times the Friedmann equation controlling the cosmological expansion (contraction) coincides with an universal Cardy formula for the entropy of the CFT on the brane. If the above is true, it seems surprising that the Friedmann equation contains information about thermodynamics of the CFT. However, the Friedmann equation at the position of the horizon is obscure because the embedding of the AdSS$_5$ spacetime to the moving domain wall is apparently singular at these points.

In this paper, we resolve this embedding problem of an AdSS$_5$-black hole spacetime into the moving domain wall by introducing a radially infalling brane (RIB), where the same problem occurs. It turns out that in the case of RIB, an embedding onto the horizon (when the brane crosses the black hole) can be defined because it belongs to a coordinate singularity.

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[1]Here we use the term "moving domain wall" loosely to refer to any 3-brane moving in 5 dimensions.

[2] For the holographic entropy bound in cosmology, one may consider either a universe-size black hole with $\ell = r_+$ or the large black hole with $\ell < r_+$ [10]. But for these cases, one cannot choose an appropriate embedding for obtaining the moving domain wall (brane). Hence we do not consider these black holes.

Similarly we show that an embedding into the MDW at the horizon is possible. We can define the Friedmann equation at these moments. Hence we can introduce a relation of the CFT/FRW-cosmology on the brane.

For cosmological embedding, let us start with an AdSS$_5$-spacetime [13] ,

$$ds_5^2 = g_{MN}dx^M dx^N = -h(r)dt^2 + \frac{1}{h(r)}dr^2 + r^2 \left[d\chi^2 + f_k(\chi)^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \quad (1)$$

where $k = 0, \pm 1$. $h(r)$ and $f_k(\chi)$ are given by

$$h(r) = k - \frac{m}{r^2} + \frac{r^2}{\ell^2}, \quad f_0(\chi) = \chi,\ f_1(\chi) = \sin\chi,\ f_{-1}(\chi) = \sinh\chi. \quad (2)$$

In the case of $m = 0$, we have an exact AdS$_5$-space. However, $m \neq 0$ generates the electric part of the Weyl tensor $E_{MP} = C_{MNPQ}n^N n^Q$ [14]. This means that the bulk spacetime has an small black hole ($\ell > r_+$) with the horizon at $r = r_+$, $r_+^2 = \ell^2(\sqrt{k^2 + 4m/\ell^2} - k)/2$ [6]. Hereafter we neither consider the universe-size ($\ell = r_+$) nor large ($r_+ > \ell$) black holes because one cannot define their embedding into the moving domain wall [10].

II. MOVING DOMAIN WALL (MDW)

Now we introduce the radial location of a MDW in the form of $r = a(\tau), t = t(\tau)$ parametrized by the proper time τ : $(t, r, \chi, \theta, \phi) \to (t(\tau), a(\tau), \chi, \theta, \phi)$. Then we expect that the induced metric of dynamical domain wall will be given by the FRW-type. Hence τ and $a(\tau)$ will imply the cosmic time and scale factor of the FRW-universe, respectively. A tangent vector (proper velocity) of this MDW

$$u = \dot{t}\frac{\partial}{\partial t} + \dot{a}\frac{\partial}{\partial a}, \quad (3)$$

is introduced to define an embedding properly. Here overdots mean differentiation with respect to τ. This is normalized to satisfy

$$u^M u^N g_{MN} = -1. \quad (4)$$

Given a tangent vector u_M, we need a normal 1-form directed toward to the bulk. Here we choose this as

$$n = \dot{a}dt - \dot{t}da, \quad n_M n_N g^{MN} = 1. \quad (5)$$

This convention is consistent with the Randall-Sundrum case in the limit of $m = 0$ [2]. Using either Eq.(3) with (4) or Eq.(5), we can express the proper time rate of the AdSS$_5$ time $\dot{t}$ in terms of $\dot{a}$ as

$$\dot{t} = \frac{\sqrt{\dot{a}^2 + h(a)}}{h(a)}. \quad (6)$$

From the above, it seems that $\dot{t}$ is not defined at $a = a_+$ because $h(a_+) = 0$. This also happens in the study of static black hole. Usually one introduces a tortoise coordinate

$r^* = \int h^{-1} dr$ to resolve it. Then Eq.(1) takes a form of $ds_5^2 = -h(dt^2 - dr^{*2}) \cdots$ and one finds the Kruscal extension. This means that $r = r_+$ is just a coordinate singularity. We confirm this from $R_{MNPQ}R^{MNPQ} = 40/\ell^4 + 72m^2/r^8$, which shows that $r = 0$ $(r = r_+)$ are true (coordinate) singularity. Our dynamic situation is different from the static case. Here a convenient coordinate is not a tortoise one r^* but r itself. Eq.(6) takes an alternative form of $h\dot{t} = \sqrt{\dot{a}^2 + h}$, which implies that $\sqrt{\dot{a}^2 + h} = 0 \to \dot{a} = 0$ at $a = a_+$. Let us study this point more carefully. An explicit form of our tangent vector is given by

$$u^M = (\frac{\sqrt{\dot{a}^2 + h(a)}}{h(a)}, \dot{a}, 0, 0, 0), \qquad u_M = (-\sqrt{\dot{a}^2 + h(a)}, \frac{\dot{a}}{h(a)}, 0, 0, 0). \tag{7}$$

On the other hand, the normal vector takes the form

$$n^M = (-\frac{\dot{a}}{h(a)}, -\sqrt{\dot{a}^2 + h(a)}, 0, 0, 0), \qquad n_M = (\dot{a}, -\frac{\sqrt{\dot{a}^2 + h(a)}}{h(a)}, 0, 0, 0). \tag{8}$$

As is emphasized again, two vectors which are essential for the embedding are well defined everywhere, except $r = r_+$. But these look like singular vectors at the horizon. Hence two moments when the brane crosses the past (future) event horizons are singular points where one may not define the moving domain wall. This persists in deriving the 4D intrinsic metric and the extrinsic curvature. The first two terms in Eq.(1) together with Eq.(6) leads to

$$-h(r)dt^2 + \frac{1}{h(r)}dr^2 \quad \to \quad -(h(a)\dot{t}^2 - \frac{\dot{a}^2}{h(a)})d\tau^2 = -d\tau^2. \tag{9}$$

Here one may worry about this connection when $h(a_+) = 0$. The 4D induced line element is

$$\begin{aligned} ds_4^2 &= -d\tau^2 + a(\tau)^2 \left[d\chi^2 + f_k(\chi)^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] \\ &\equiv h_{\mu\nu} dx^\mu dx^\nu, \end{aligned} \tag{10}$$

where we use the Greek indices only for the brane. Actually the embedding of an AdSS$_5$ space to the FRW-universe is a $2(t, r) \to 1(\tau)$-mapping. The projection tensor is given by $h_{MN} = g_{MN} - n_M n_N$ and its determinant is zero. Hence its inverse h^{MN} cannot be defined. This means that the above embedding belongs to a peculiar mapping to obtain the induced metric $h_{\mu\nu}$ from the AdSS$_5$ black hole spacetime g_{MN} with n_M. In addition, the extrinsic curvature is defined by

$$K_{\tau\tau} = K_{MN}u^M u^N = (h(a)\dot{t})^{-1}(\ddot{a} + h'(a)/2) = \frac{\ddot{a} + h'(a)/2}{\sqrt{\dot{a}^2 + h(a)}}, \tag{11}$$

$$K_{\chi\chi} = K_{\theta\theta} = K_{\phi\phi} = -h(a)\dot{t}a = -\sqrt{\dot{a}^2 + h(a)}\ a, \tag{12}$$

where prime stands for derivative with respect to a. We observe that $K_{\tau\tau}$ looks like ill-defined as $\frac{\ddot{a}+h'(a)/2}{0}$, and $K_{\chi\chi} = K_{\theta\theta} = K_{\phi\phi} = 0$ at $a = a_+$. As we will see later, this belongs to apparent phenomena. A localized matter on the brane implies that the extrinsic curvature jumps across the brane. This jump is described by the Israel junction condition

$$K_{\mu\nu} = -\kappa^2 \left(T_{\mu\nu} - \frac{1}{3} T^\lambda_\lambda h_{\mu\nu} \right) \tag{13}$$

with $\kappa^2 = 8\pi G_5^N$. For cosmological purpose, we may introduce a localized stress-energy tensor on the brane as the 4D perfect fluid

$$T_{\mu\nu} = (\varrho + p)u_\mu u_\nu + p\, h_{\mu\nu}. \tag{14}$$

Here $\varrho = \rho + \sigma$ ($p = P - \sigma$), where ρ (P) is the energy density (pressure) of the localized matter and σ is the brane tension. In the case of $\rho = P = 0$, the r.h.s. of Eq.(13) leads to a form of the RS case as $-\frac{\sigma\kappa^2}{3}h_{\mu\nu}$. From Eqs.(13), one finds the space component of the junction condition

$$\sqrt{h(a) + \dot{a}^2} = \frac{\kappa^2}{3}\sigma a. \tag{15}$$

For a single AdSS$_5$, we have the fine-tuned brane tension $\sigma = 3/(\kappa^2\ell)$. The above equation leads to

$$H^2 = -\frac{k}{a^2} + \frac{m}{a^4}, \tag{16}$$

where $H = \dot{a}/a$ is the expansion rate. The term of m/a^4 originates from the electric (Coulomb) part of the 5D Weyl tensor, $E_{00} \sim m/a^4$ [14,11]. This term behaves like radiation [4]. Especially for $k = 1$, we have $m = \frac{16\pi G_5^N M}{3V(S^3)}, M = \frac{a}{\ell}E, V = a^3 V(S^3), G_5^N = \frac{\ell}{2}G_4^N$. Then one finds a CFT-radiation dominated universe

$$H^2 = -\frac{1}{a^2} + \frac{8\pi G_4^N}{3}\rho_{CFT}, \quad \rho_{CFT} = \frac{E}{V}. \tag{17}$$

It seems that the equation (15) is well-defined even at $a = a_+$. Thus this leads to $H = \pm 1/\ell$ at $a = a_+$, which is just the case mentioned in ref. [12]. At this stage, this point is not clear. From Eq.(6), one finds $\dot{a}^2 = 0$ at $a = a_+$. This means that $H^2a^2 = 0 \rightarrow H = 0$ at $a = a_+$ because of $a_+ \neq 0$. Naively we find a contradiction. Also, considering the extrinsic curvature expressed in terms of $a, h(a)$, the junction condition may not be defined at $a = a_+$. To resolve this problem, we introduce a radially infalling brane, where the same situation occurs as in the MDW picture.

III. RADIALLY INFALLING BRANE (RIB)

A 3-brane action is usually given by the Nambu-Goto action. Here for cosmological purpose, we consider its point particle limit of brane $\rightarrow$ body. The corresponding action with unit mass on the AdSS$_5$ background space [15] is given by

$$\mathcal{L} = -\frac{1}{2}g_{MN}\frac{dx^M}{d\tau}\frac{dx^N}{d\tau} = \frac{1}{2}(h\dot{t}^2 - \frac{\dot{a}^2}{h} + \cdots) \tag{18}$$

where $\cdots$ means the angular part. This part is not relevant to our purpose because we consider only radial time-like geodesics.

For time-like geodesics, τ is proper time of the RIB describing the geodesic. The corresponding canonical momenta are

$$p_t = \frac{\partial \mathcal{L}}{\partial \dot{t}} = h(a)\dot{t}, \quad p_a = -\frac{\partial \mathcal{L}}{\partial \dot{a}} = \frac{\dot{a}}{h(a)}, \cdots. \tag{19}$$

p_t (p_a) correspond to $-u_t = -n^a$ ($u_a = -n^t$) in the MDW approach. Hence the apparent singularity at $a = a_+$ appears in the RIF picture. We always choose $2\mathcal{L} = 1$ by rescaling the affine parameter τ for time-like geodesics. This is just the same as was found in the normalization condition Eq.(6) for the tangent vector u^M and normal vector n_M in the MDW approach. Here we get an integral of the motion from the fact that t is a cyclic coordinate : $\frac{dp_t}{d\tau} = \frac{\partial \mathcal{L}}{\partial t} = 0$,

$$p_t = h(a)\dot{t} = \sqrt{\dot{a}^2 + h(a)} = E \tag{20}$$

where E is a constant of the motion. The above equation corresponds to Eq.(15) in the MDW approach. On the other hand, $2\mathcal{L} = 1$ means

$$\frac{1}{h(a)}(E^2 - \dot{a}^2) = 1. \tag{21}$$

This is nothing new and corresponds to Eq.(6) in the MDW approach. Different choices of the constant E corresponds to different initial conditions. For simplicity, let us make a choice of $E = 1$, which corresponds to dropping in a RIB from infinity with zero initial velocity. In the limit of $\ell \to \infty$, the situation becomes rather clear. Hence first we consider the geodesic of RIB in the 5D Schwarzschild black hole spacetime.

A. RIB in 5D Schwarzschild space

In the case of $\ell \to \infty$, we find from Eq.(21)

$$\frac{1}{\dot{a}^2} = \frac{a^2}{m} \to \quad d\tau = -\frac{a}{\sqrt{m}}da, \tag{22}$$

where we take the negative square root because we consider an infalling body into the black hole from the large a_0 ($a_0 >> \sqrt{m}$). This leads to

$$\tau - \tau_0 = \frac{1}{2\sqrt{m}}(a_0^2 - a^2) \tag{23}$$

where the RIB is located at a_0 at proper time τ_0. Any singular behavior does not appear at the Schwarzschild radius $a = a_+ = \sqrt{m}$ and the RIB falls continuously to $a = 0$ in a finite proper time. But if we describe the motion of RIB in terms of the Schwarzschild coordinates (t, a), then

$$\frac{dt}{da} = \frac{\dot{t}}{\dot{a}} = -\frac{a}{\sqrt{m}h(a)} \tag{24}$$

which is integrated as

$$t - t_0 = \frac{1}{2\sqrt{m}}[a_0^2 - a^2 + m\log\frac{(a_0 - \sqrt{m})(a_0 + \sqrt{m})}{(a - \sqrt{m})(a + \sqrt{m})}]. \tag{25}$$

In the limit of $a \to a_+ = \sqrt{m}$, one has

$$t - t_0 \simeq -\frac{\sqrt{m}}{2} \log(a - \sqrt{m}) \to \infty \tag{26}$$

which means that $a_+ = \sqrt{m}$ is approached but never passed. The coordinate t is useful and physically meaningful asymptotically at large a since it corresponds to proper time measured by an observer at rest far away from the origin (that is, $dt = d\tau$ when $a \to \infty$). From the point of view of such an observer, it takes an infinite time for the RIB to reach $a = a_+$. On the other hand, from the point of view of the RIB itself, it reaches $a = a_+$ and $a = 0$ within finite proper time. Clearly, the Schwarzschild time coordinate t is inappropriate for describing a radially infalling motion.

B. RIB in AdSS$_5$ space

Now we are in a position to study the motion of RIB in the AdSS$_5$ black hole spacetime. In this case, we have

$$\frac{1}{\dot{a}^2} = \frac{a^2\ell^2}{m\ell^2 - a^4} \to \quad d\tau = -\frac{a\ell}{\sqrt{m\ell^2 - a^4}} da \tag{27}$$

which leads to

$$\tau - \tau_0 = \frac{\ell}{2}\left(\tan^{-1}\left[\frac{a_0^2}{\sqrt{m\ell^2 - a_0^4}}\right] - \tan^{-1}\left[\frac{a^2}{\sqrt{m\ell^2 - a^4}}\right]\right) \leq \frac{\pi\ell}{4}. \tag{28}$$

Hence it takes finite proper time for a RIB to reach $a = a_+$ and $a = 0$. In order to a simple relation between τ and t, let us consider the asymptotic form of AdSS$_5$ space with $k = 1$:$\lim_{a\to\infty}[\frac{\ell^2}{a^2} ds_5^2] = -dt^2 + \ell^2 d\Omega_3^2$. We find that the proper time τ is equal to the AdSS$_5$ time t only when the radius of S^3 is set to be ℓ. For a finite a, the relation becomes quite complicated. Here we have

$$\frac{dt}{da} = \frac{\dot{t}}{\dot{a}} = -\frac{a\ell}{\sqrt{m\ell^2 - a^4}} \frac{1}{h(a)}. \tag{29}$$

We note that integration of Eq.(29) leads to a complicated from for $t - t_0$. This is related to the presence of a term a^2/ℓ^2 in $h(a)$. Hence the proper time τ is an affine variable which describes time-like geodesic (the motion of RIB) correctly in AdSS$_5$ space. On the other hand, it turns out that the AdSS$_5$ time coordinate t is not appropriate for describing the RIB which falls into the Schwarzschild black hole in anti de Sitter space.

IV. DISCUSSION

First let us compare the MDW with the RIB. The equation of MDW with $k = 1$ is given by

$$\frac{1}{2}\dot{a}^2 + V(a)_{MDW} = -\frac{1}{2} \tag{30}$$

with its potential $V(a)_{MDW} = -\frac{m}{2a^2}$. This corresponds to the motion of a point particle with unit mass and a negative energy rolling in a potential V_{MDW}. Actually the scale factor increases from $a = 0$ to maximum size, $V(a_{max})_{MDW} = -1/2$, and recollapses into $a = 0$. On the other hand, the motion of RIB is given by

$$\frac{1}{2}\dot{a}^2 + V(a)_{RIB} = 0 \tag{31}$$

with a potential $V(a)_{RIB} = -\frac{m}{2a^2} + \frac{a^2}{2\ell^2}$. This corresponds to the equation for a particle of unit mass and zero energy rolling in a potential V_{RIB}. Hence it corresponds to a radially infalling body starting at $a_0 >> a_+$. Hence it is suggested that the motion of RIF covers half that of MDW. For $k = 0$ case, the MDW takes the same equation as in the RIB with a slightly different potential. But this does not make a significant change. In this case, the MDW starts at $a = 0$ and ends up at $a = \infty$ [11]. As a reverse process, the RIB starts at $a = \infty$ and ends up at $a = 0$. Furthermore, at the horizons, the expansion rate of MDW is $H_{MDW} = \pm 1/\ell$ whereas that of RIB is $H_{RIB} = \pm 1/a_+$.

However an important difference is that $H^2_{RIB} = -1/\ell^2 + m/a^4$ is not a kind of Friedmann-like equations. The reason is that as is shown in $H^2_{RIB} = \cdots$, the first term of $1/\ell^2$ means that the size of the background $AdSS_5$ space where the RIB moves is always fixed as the spatial curvature of the $AdSS_5$ (ℓ). On the other hand, the Friedmann-like equation (17) for the MDW means that at each instant the size of universe is given by the spatial curvature (scale factor :a).

Anyway, it is clear from the analysis of the RIB that a genuine coordinate for the MDW is not the $AdSS_5$ time coordinate t but the proper time τ. Finally it turns out that the apparent singular behaviors of the normal, tangent vectors, and extrinsic curvature at the horizon belong to coordinate artifacts. Hence Eq.(17) implies a radiation-dominated universe of $\rho \sim a^{-4}$. This radiation can be identified with the finite temperature CFT that is dual to the $AdSS_5$ geometry. This prescription is valid when the MDW crosses the horizons. Thanks to this , an universal Cardy formula for the entropy of the CFT can be given by the Friedmann equation at the horizon [12].

In conclusion, the moving domain wall approach provides us a nice tool to study the brane cosmology including the location of the horizon.

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